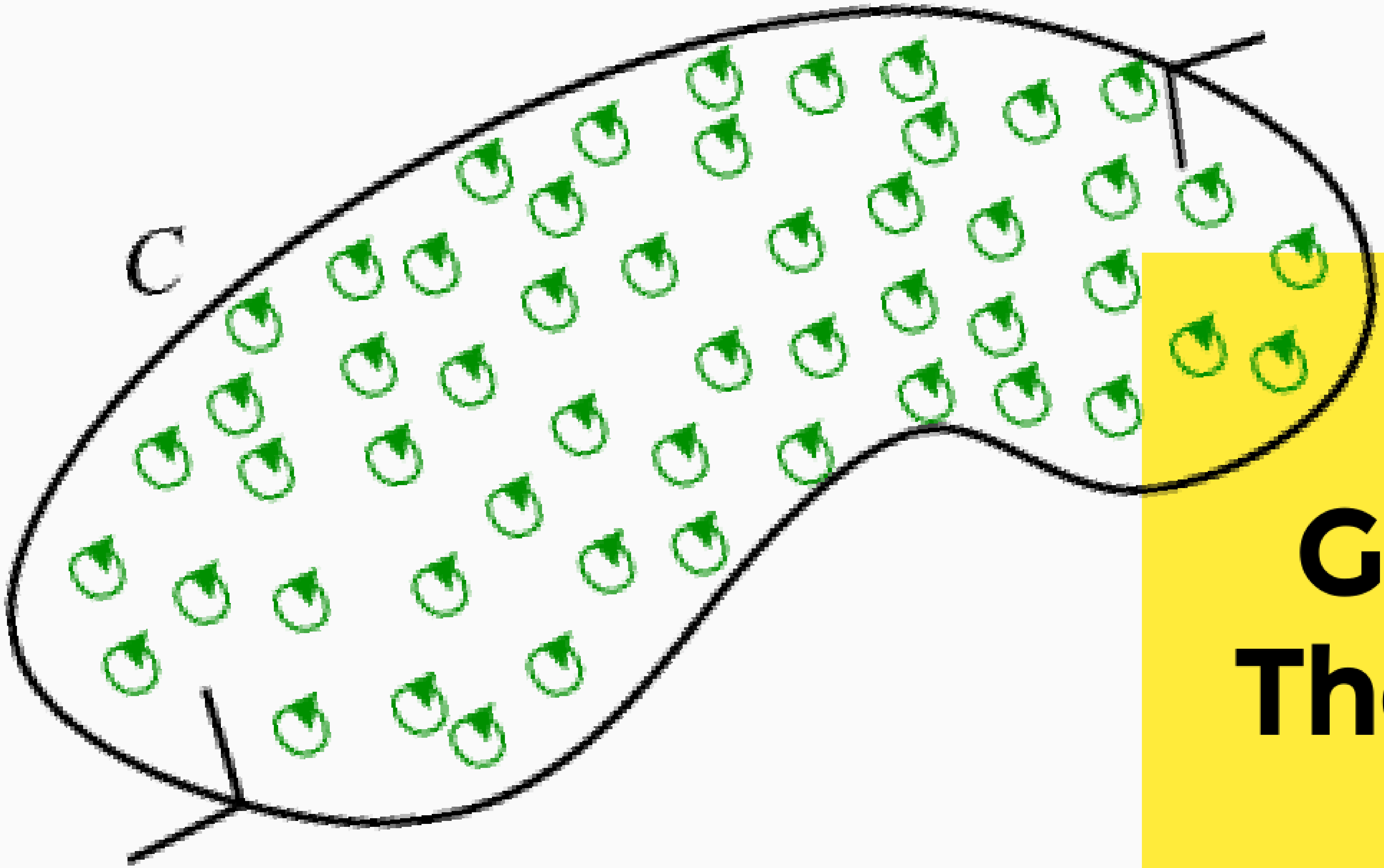
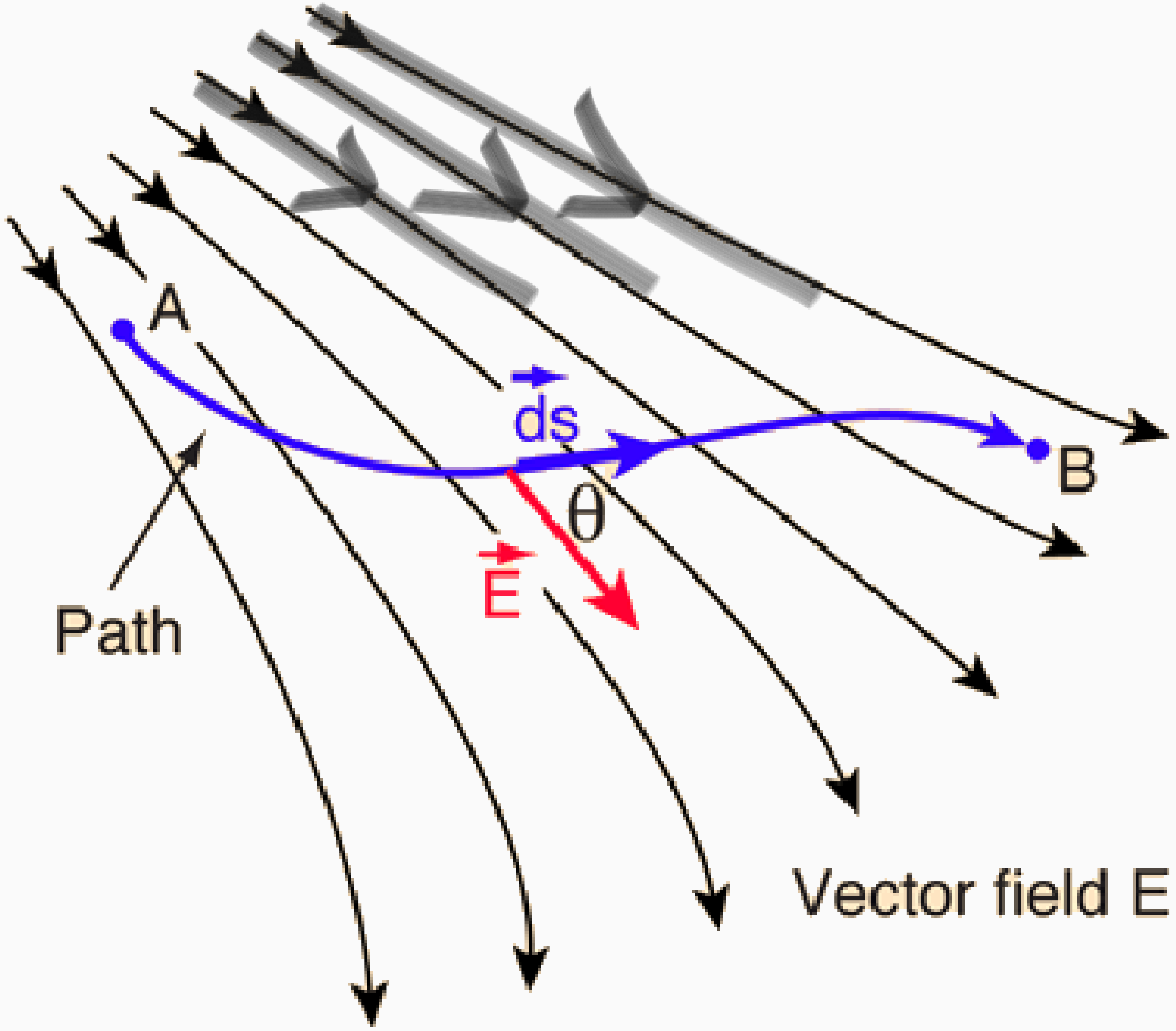


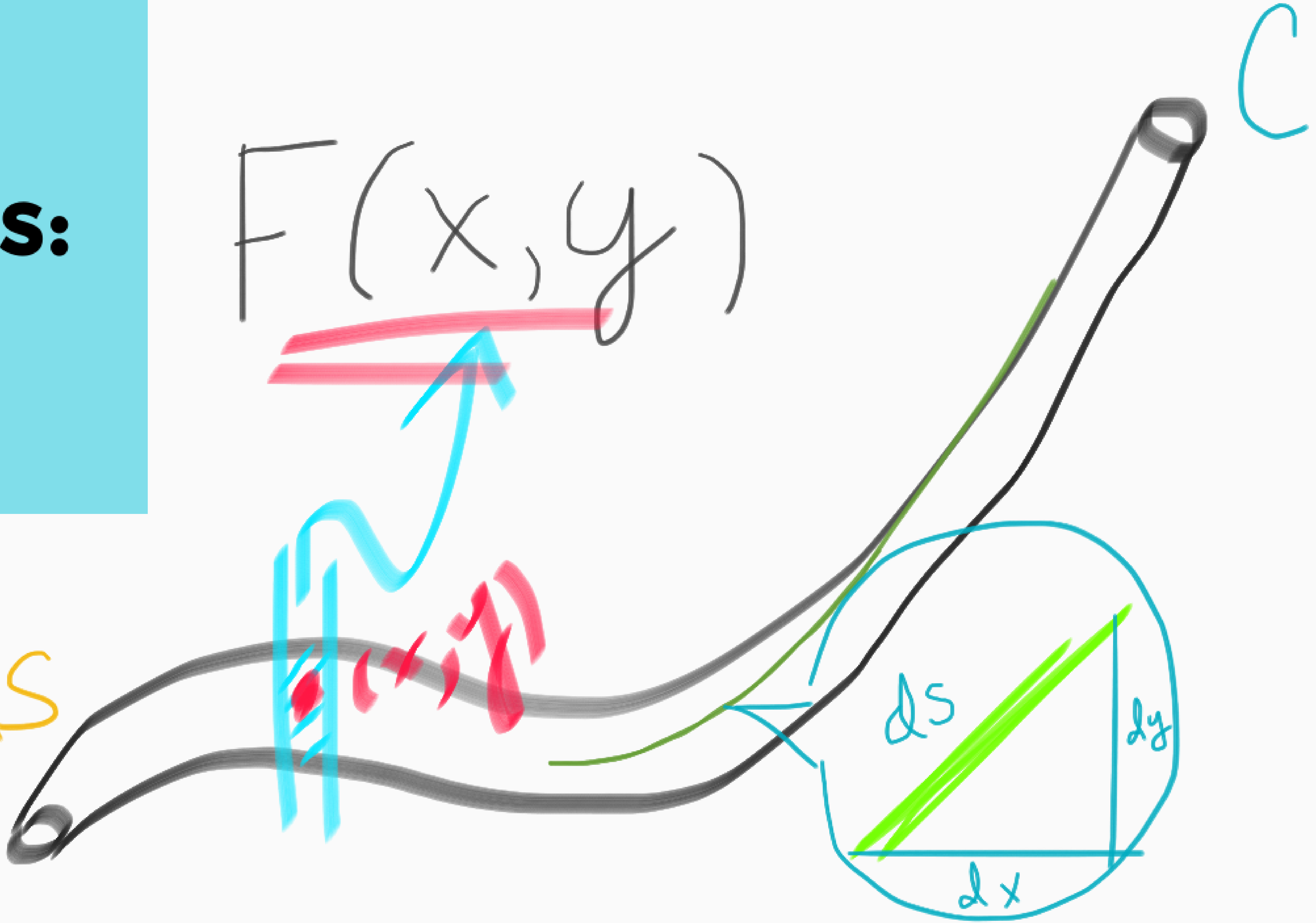
Green's Theorem!

Line Integrals



Line Integrals: Mass!

$$\int_C F(x,y)\,ds$$



Line Integrals: Mass!

$$ds = \sqrt{dx^2 + dy^2}$$

$$\int_C F(x,y)\sqrt{dx^2+dy^2}$$

$$\int_C F(x,y)\,ds \quad \int_C F(x(t),y(t))\sqrt{dx^2+dy^2}$$

Line Integrals: Mass!

$$\int_C F(x(t),y(t))\frac{dt}{dt}\sqrt{dx^2+dy^2}$$

$$\int_C F(x(t),y(t))\sqrt{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}}\,dt$$

let $\vec{r}(t)=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$

$$\int_C F(\vec{r}(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

LINZ $\int$:

- Mass
- Work

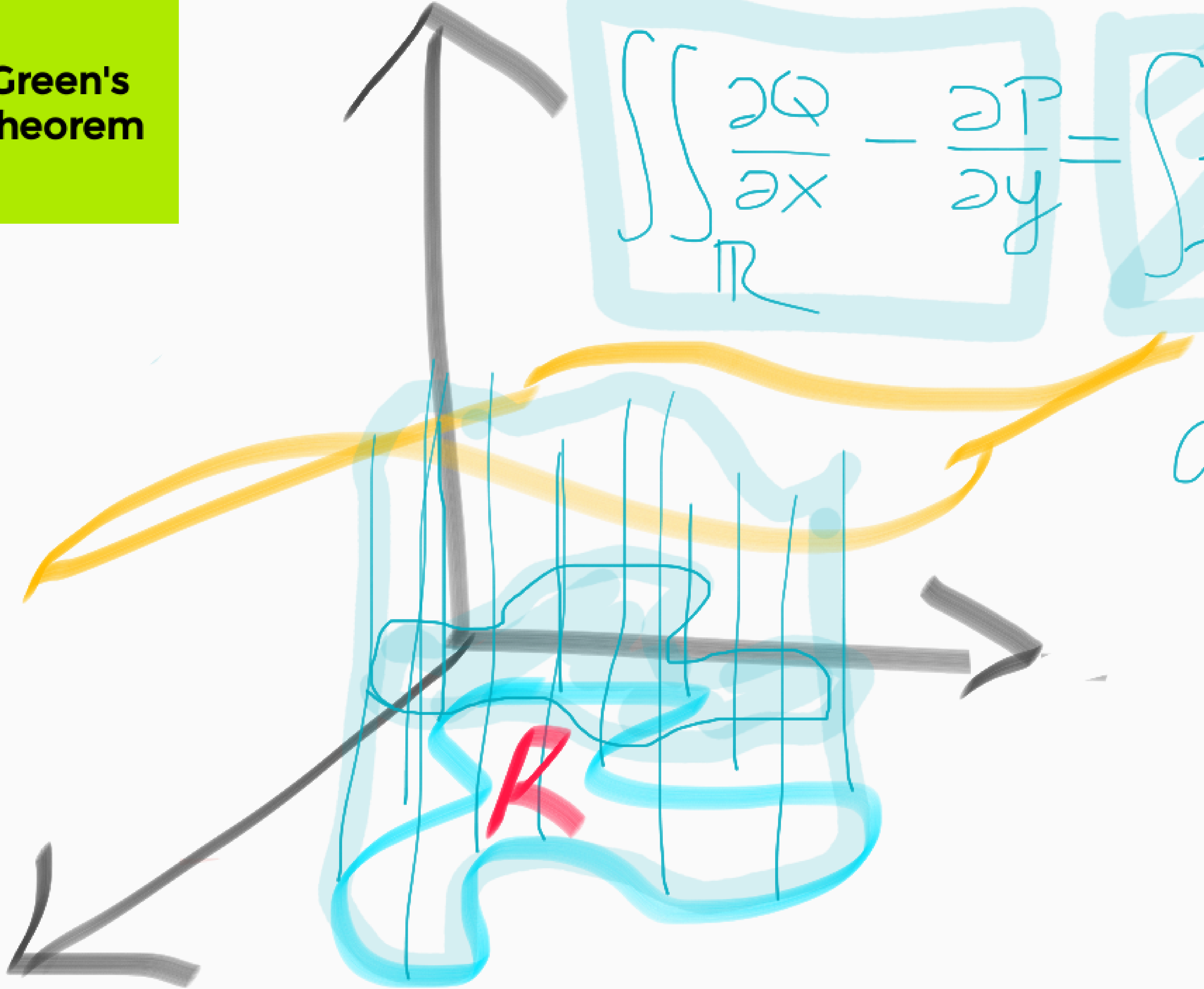
Green's Theorem

line

$$\oint_C \vec{F} \cdot \hat{r}\, ds = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Green's Theorem

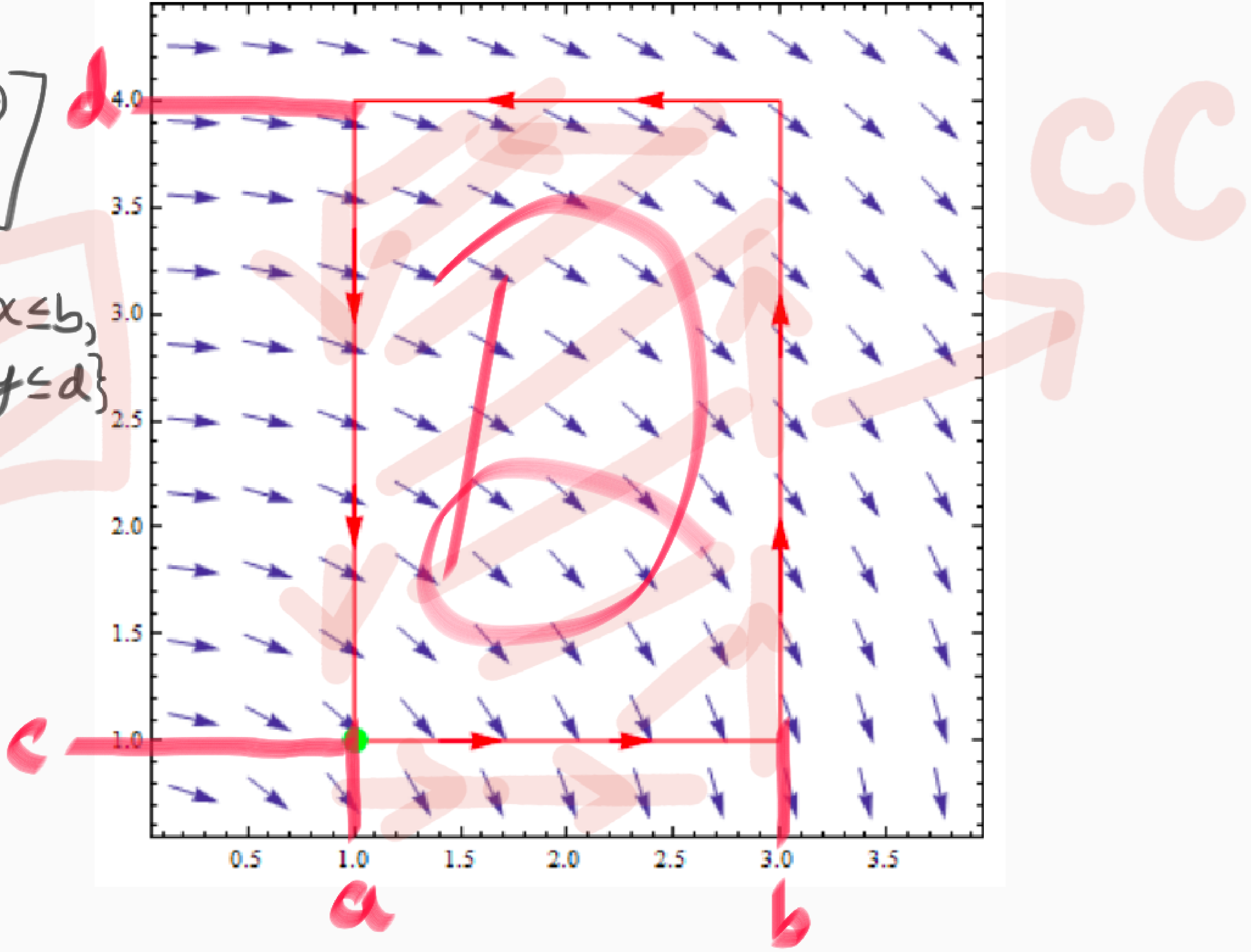
$$\boxed{\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}} = \boxed{\int_R F \cdot \hat{r}\, ds}$$

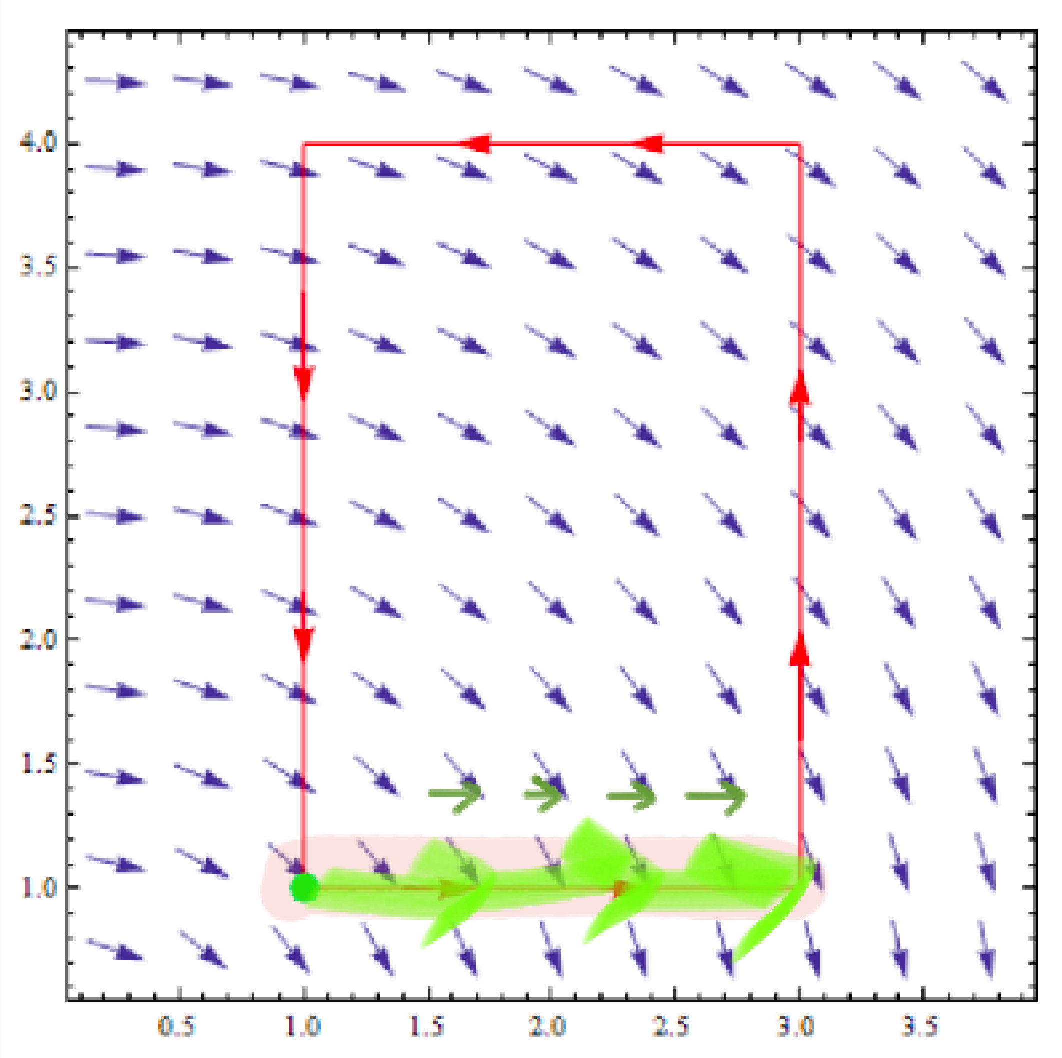


$$d = \frac{m}{v} \quad \frac{kg}{m^3}$$

$$F(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

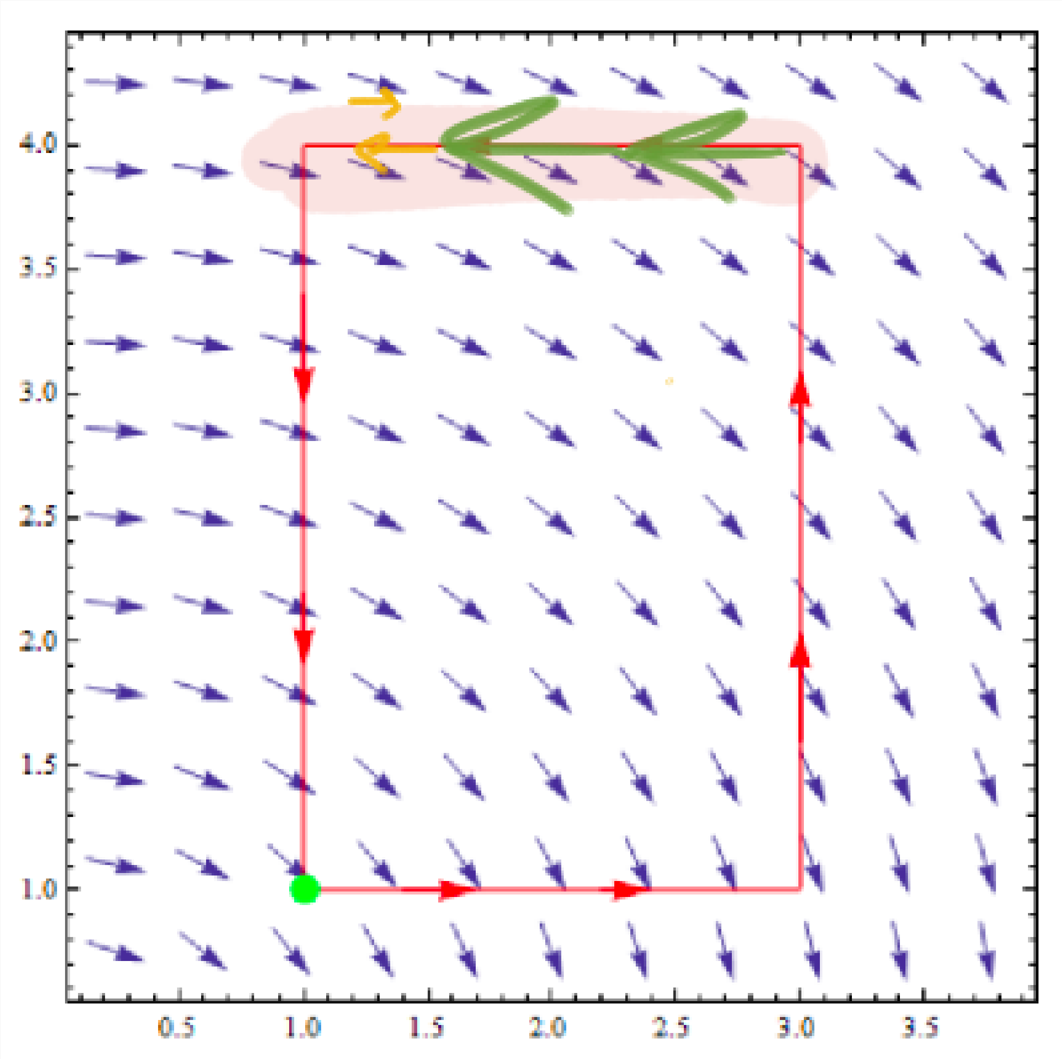
$$D = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$





$$\int_C F \cdot \hat{r}\, ds \qquad \boxed{\int_C P(x,c)\, ds}$$

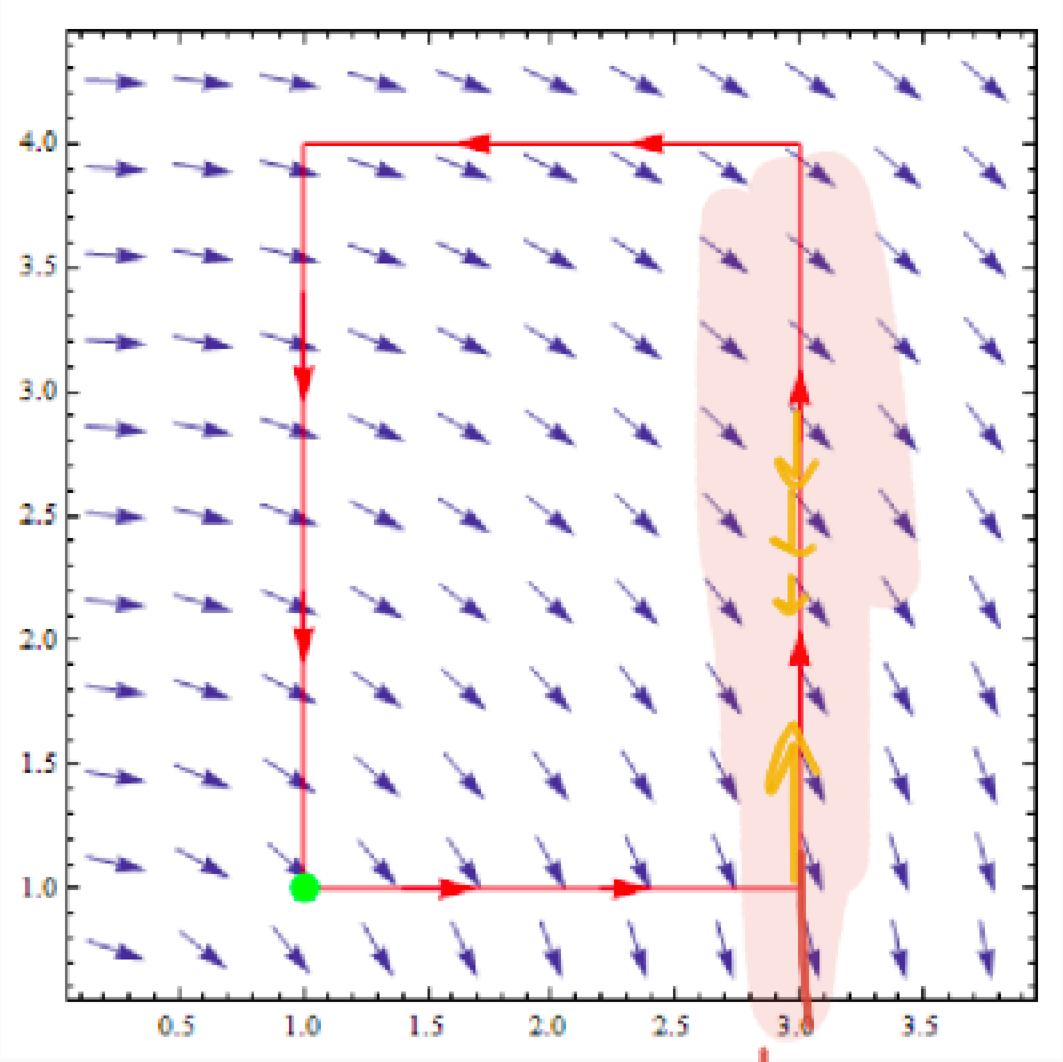
$$\hat{r} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \end{bmatrix} = P$$



$$\int F\cdot\hat{r}\,ds = \int_C -P(x,d)\,ds$$

$$F\cdot\hat{r} = \begin{bmatrix}P\\Q\end{bmatrix}\cdot\begin{bmatrix}-1\\0\end{bmatrix} = -P$$

$$\boxed{\int_C -P(x,d)\,ds}$$

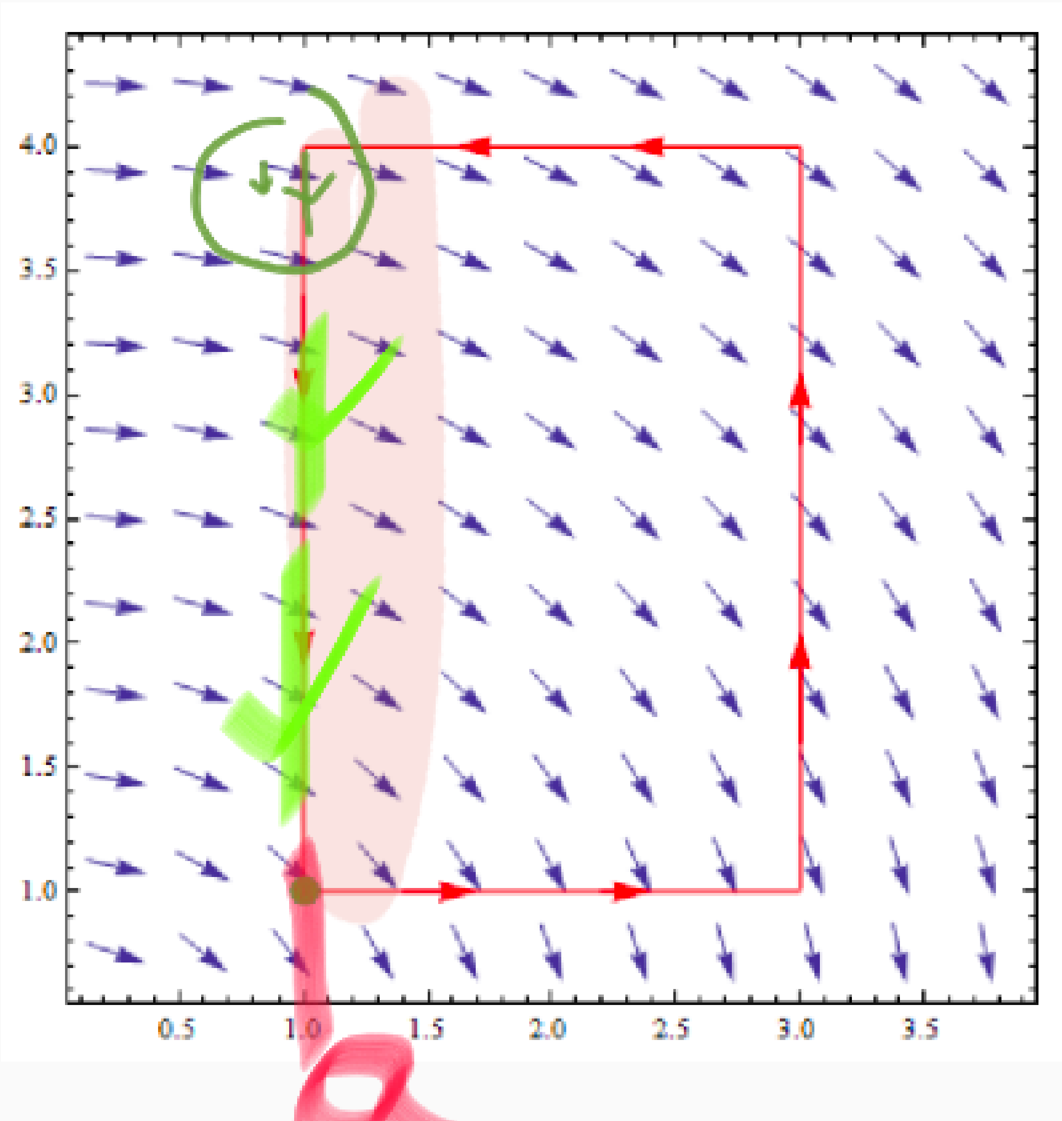


b

$$\boxed{-\int Q(b,y)\,ds}$$

$$\int F \cdot \hat{r}\,ds$$

$$F \cdot \hat{r} = \begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = Q$$



$$F \circ \hat{r} = \begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -Q$$

$$\int_C F \circ \hat{r}\, ds = -\boxed{\int_C Q(a,y)\, ds}$$

$$\boxed{-\int Q(b,y)\,ds} + \boxed{\int_C Q(a,y)\,ds}$$

$$\int_C Q(a,y) - Q(b,y)\,ds$$

$$Q(a,y) - Q(b,y) = \int_b^a Q(x,y)$$

$$\int_a^b f(x) = F(a) - F(b)$$

$$\int_C Q(a,y) - Q(b,y)\, ds$$

$$Q(a,y) - Q(b,y) = \int_b^a \frac{\partial Q}{\partial x}\, \partial x$$

$$\int_c \int_a^b \frac{\partial Q}{\partial x}\,\partial x - \frac{\partial P}{\partial y}\,\partial y$$

$$\int_C Q(a,y) - Q(b,y)\,ds$$

$$Q(a,y) - Q(b,y) = \int_b^a \frac{\partial Q}{\partial x}\,\partial x$$

$$\int_c \int_a^b \frac{\partial Q}{\partial x} \partial x - \frac{\partial P}{\partial y} \partial y$$

$$\int_c \int_a^b \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \partial x \partial y$$

$$\iint (\partial d \text{ curl } \nabla \times F) \partial x \partial y$$ //

Problem 1. Let C be the perimeter of the rectangle with sides $x = 1$, $y = 2$, $x = 3$, and $y = 3$. Evaluate the integral

$$\int_C (3x^4 + 5)\,dx + (y^5 + 3y^2 - 1)\,dy.$$

$\vec{F} \cdot \hat{r}$

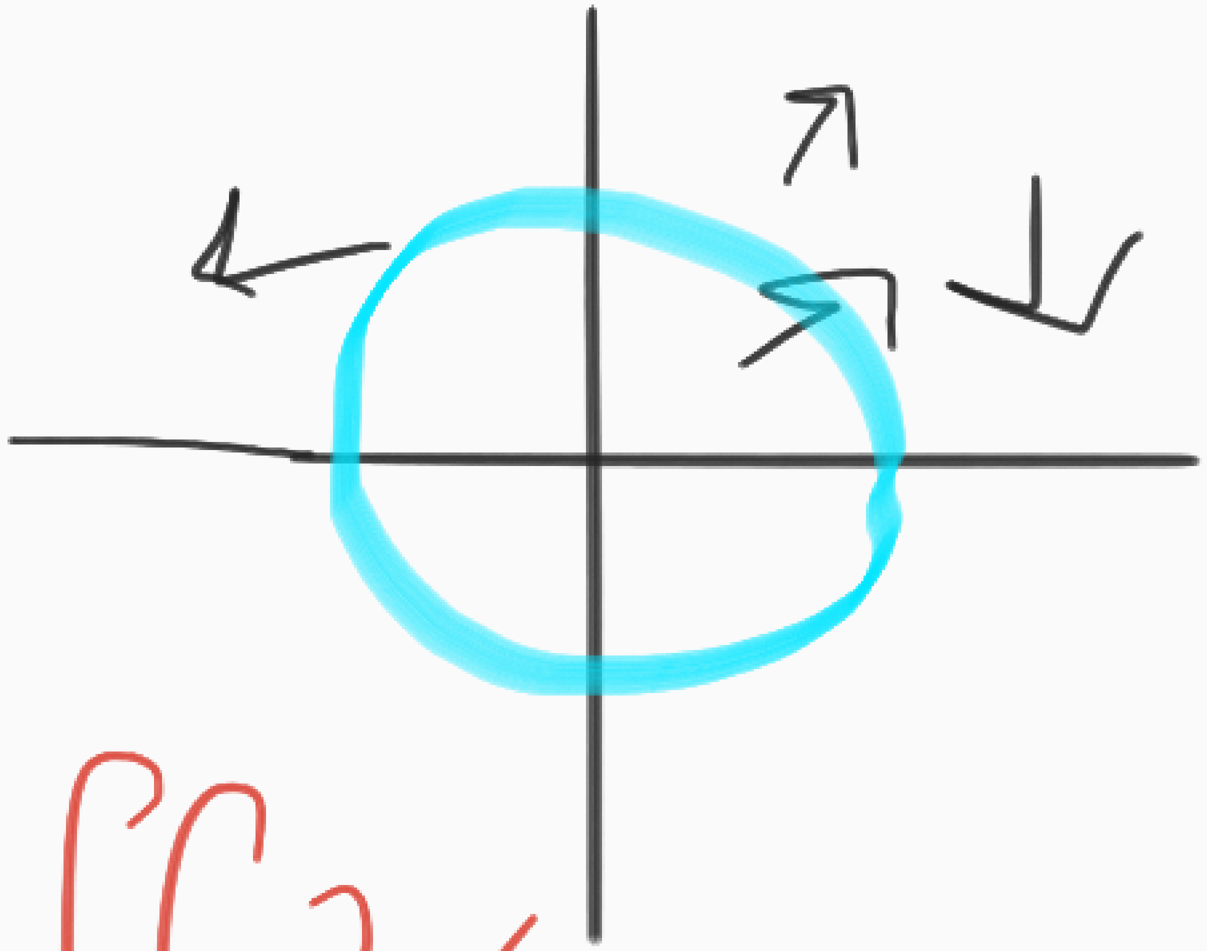
$$\oint_C \vec{F} \cdot \hat{r}\, ds = \int_c^d \int_a^b \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, \partial x\, \partial y$$

$$= \int_c^d \int_a^b \frac{\partial}{\partial x}\left(y^5 + 3y^2 - 1\right) - \frac{\partial}{\partial y}\left(3x^4 + 5\right) dx\,dy$$

$$= \boxed{0}$$

Problem 2. Let $\boldsymbol{F}(x,y) = (\underbrace{2y + e^x}_{P})\hat{\boldsymbol{i}} + (\underbrace{x + \sin(y^2)}_{Q})\hat{\boldsymbol{j}}$ and C be the circle $x^2 + y^2 = 1$. Evaluate

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s}.$$



$$\int \vec{F} \cdot ds = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \partial x \partial y$$

$$\iint \frac{\partial}{\partial x}(\underbrace{x+\sin(y^2)}_{1}) - \frac{\partial}{\partial y}(\underbrace{2y+e^x}_{2}) \underbrace{\partial x \partial y}_{dA}$$

Problem 2. Let $\boldsymbol{F}(x, y) = (2y + e^x)\hat{\boldsymbol{i}} + (x + \sin(y^2))\hat{\boldsymbol{j}}$ and C be the circle $x^2 + y^2 = 1$. Evaluate

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s}.$$

P Q

$$\iint -1\, dA = -1 \iint dA = -1A = -\pi r^2 = \boxed{-\pi}$$

$$\iint \frac{\partial}{\partial x}\underbrace{\left(x+\sin(y^2)\right)}_{1} - \frac{\partial}{\partial y}\underbrace{\left(2y+e^x\right)}_{2} \underbrace{\partial x \partial y}_{dA}$$

Problem 3. Use Green's Theorem to find the area of a disc of radius a.

$$\int_C x\,dy - y\,dx = \int \left(\begin{matrix} a\cos\theta \cdot \\ a\cos\theta\, dt \end{matrix}\right) - \left(\begin{matrix} a\sin\theta \cdot \\ -a\sin\theta\, dt \end{matrix}\right)$$

$$= \int (a^2\cos^2\theta\, dt) + (a^2\sin^2\theta\, dt)$$

$$x = a\cos\theta \Rightarrow dx = -a\sin\theta\, dt$$

$$y = a\sin\theta \Rightarrow dy = a\cos\theta\, dt$$

$$= \int_0^{2\pi} a^2 \underbrace{(\sin^2\theta + \cos^2\theta)}_{1}\, dt$$

$$= a^2 t\Big|_0^{2\pi} \Rightarrow (2\pi a^2)\tfrac{1}{2} = \boxed{\pi a^2}$$